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Minimal distance from a point to n lines

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Resumen

We present a solution to the problem of computing a point in the plane minimizing the distance to n given lines. We used an experimental mathematics approach: using dynamic geometry software, we gathered data which allowed us to conjecture and testing our hypothesis. Finally we formalized our reasoning to prove the conjecture. Although this solution was known, our method could be of interest.

Palabras y frases claves: Minimal distance, experimental mathematics.

Clasificación de la AMS: 51K99

1 Introduction

In this work we describe an experimental mathematics strategy (see [1]) to solve a geometric problem : we use dynamic geometry software to gather data, state and check conjectures, and then look for formal proofs of the conjectures that survives the experimental phase. We use the software CABRI II plus, which allows us to represent the problem and perform calculations. The dragging mode, characteristic of dynamic geometry software (see [2]), allows us to check our conjectures in a huge number cases. Although our solution is not new (see [5]), this experimental strategy could be of interest in research and mathematical education.

The problem solved is the following:

Problem 1 (MIN). *Given n lines, determine a point minimizing the sum of the distances from this point to the n given lines.*

From now on we will use the symbol MIN to denote the above problem.

Organization of the work. The paper is organized into two sections. Section one deals with the experimental work and the search for a suitable conjecture encompassing the experimental results. In section two we propose a conjecture and we prove it.

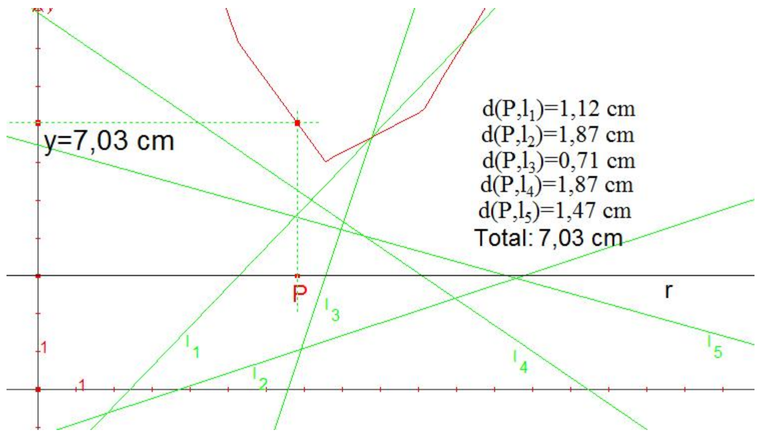

Figure 1: First exploration.

2 The problem and the experimental exploration

In this section we describe the experimental work carried out with CABRI II plus.

We asked us: Which is the structure of the solution set of a typical instance of the problem MIN? Initially, we considered configurations constituted by five lines, that is: we considered instances of the problem MIN, determined by no more than five lines. We used CABRI II plus to find the solution set of the instances considered, we looked for regularities arising in the computed solution sets, and then we used this information to formulate corresponding conjectures. Once a conjecture was proposed, we checked its soundness using new instances of the basic problem.

Experimental work (with 5 lines)

In the experimental work we planed to move a point around the plane and calculate the sum of its distances to the given lines in order to find some properties which could lead us to a conjecture.

In the following experimental device, we put a point on a line, so we can drag the point on the line to reach all positions on it, and drag the line to reach all positions on the plane, and with the locus tool we represent the function of the sum of the distances to the given lines.

1. We drew a horizontal line r with a point P on it. Then, we dragged the point along the line, getting in this way all possible abscissas. After that we began to move the line itself, without changing its direction, getting in this way all the possible ordinates.

2. We measured the distances from P to the five given lines, we added those distances, and transferred this sum to the y axis (see Figure 1).

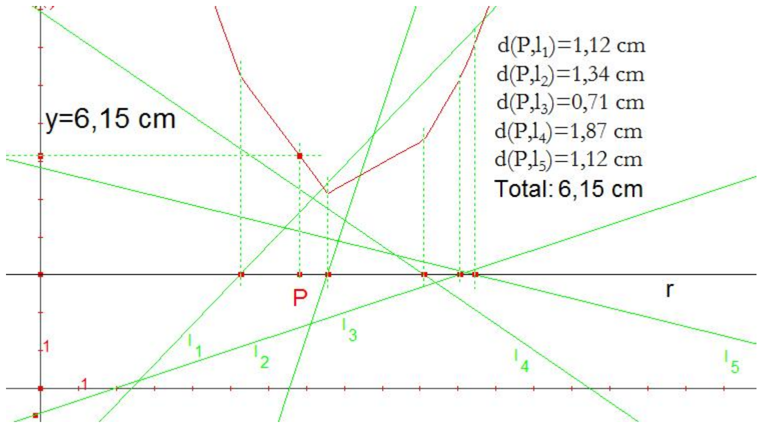

Figure 2: Second exploration.

3. We constructed a point Q whose abscissa is the abscissa of P and whose ordinate is the sum of the distances from P to the five given lines.

4. We drew the locus of Q with respect to P, and we obtained a polygonal line with a finite number of vertices, the abscissas of those vertices matched the abscissas of the intersection points of the horizontal line and the five given lines (see Figure 2).

5. We observed that one of those vertices had minimal ordinate.

6. We activated the trace of the locus, dragged the horizontal line, and we saw that there was one point for which the sum of distances to the five given lines was minimal (see Figure 3). Then, we observed that the point minimizing the distance belonged to the intersection of pairs of the five given lines.

7. We repeated the process many times, that is: we consider many configurations of lines and we used CABRI II plus to compute the solution-sets of each one of the configurations considered (see Figure 4). We observed that, for each one of the experiments performed, there was one point in the computed solution-set that belonged to the intersection of the five given lines.

3 A succesful conjecture and its proof

Our conjecture is that at least one point of the solution-set belongs to the intersection of the given lines.

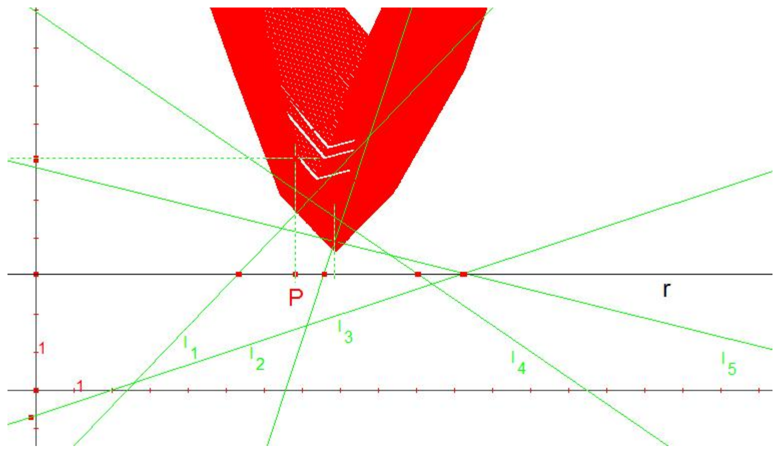

Figure 3: Third exploration.

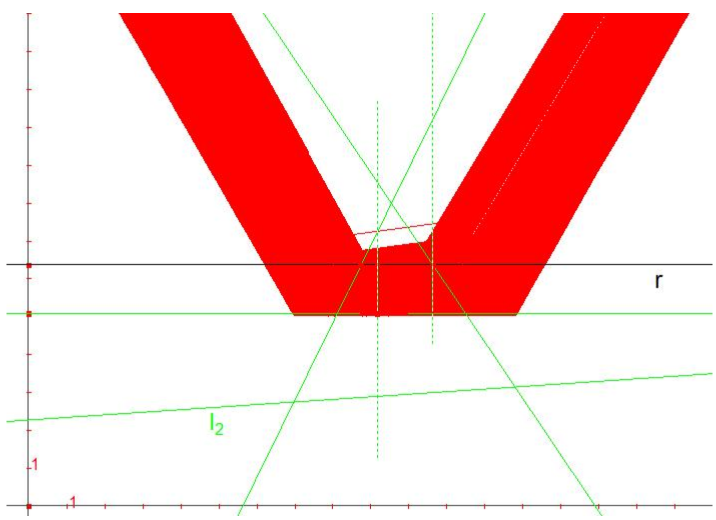

Figure 4: Fourth exploration.

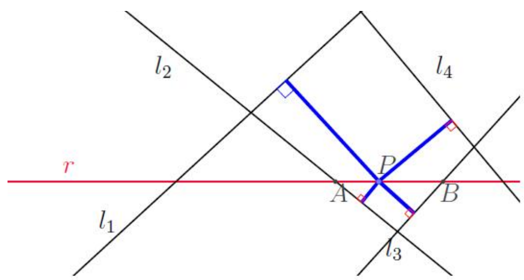

Figure 5: Perpendiculars from P to the given lines.

Proposition 1. *Suppose we have n given lines in the plane and suppose that they are not pairwise parallel.*[1] *Given P, a point in the plane, we use the symbol S_p to denote the sum of the distances from P to the n given lines. Then, we have that S_p reaches its minimum value at one of the points located on the intersection of at least two of the n given lines.*

From now on, we use the term *intersection points* to denote the points that belong to the intersection of at least two of the given lines. Before proving the above proposition we have to prove next lemma.

Lemma 1. *Let r be a line, and suppose that r meets at least one of the n given lines, suppose that $\overline{AB}$ is a segment that is located between two consecutive intersection points of r with the n given lines*[2]*, then the restriction of S_P to $\overline{AB}$ is a linear function.*

Proof. We pick a point P on $\overline{AB}$, and from P we draw perpendicular segments to the n lines, whose lengths are equal to the distances from P to the given lines (see Figure 5).

If the lines are not parallel to r, each of those orthogonal segments is a cathetus of a right triangle with hypotenuse in r, being the other cathetus a segment of the corresponding line[3] (see Figure 6).

Clearly, when P moves to P' on $\overline{AB}$, each of the right triangles defined by P is similar to the corresponding one defined by P'.

We will prove inductively that the sum of distances from P to the given lines changes proportionally when P moves on the segment $\overline{AB}$.

Proof by induction Let us take initially two lines l_1 and l_2. Let A, B be the intersection points of r with l_1 and l_2, respectively. Let $\overline{AD}$ and $\overline{BE}$ be

[1]If all lines are parallel, there is an infinity of points minimizing the distance to the lines. This case will be discussed in another paper.

[2]If r cuts exactly one line, the segment becomes a point and the graph of S_P becomes also a point.

[3]If the lines are parallel to r they do not form triangles, but their distances to P will be constants and these constants will not affect the sum.

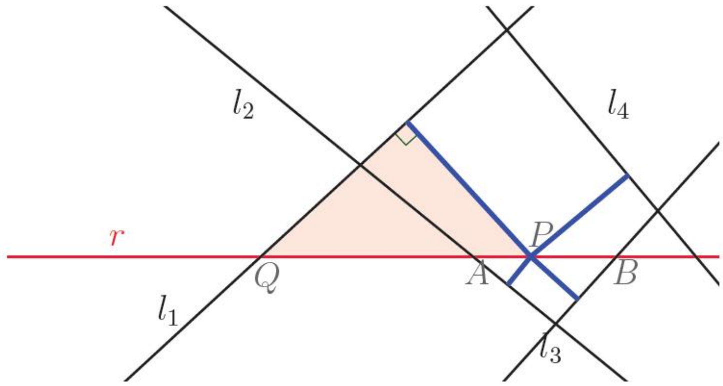

Figure 6: Right triangle with hypotenuse in r.

the orthogonal segments to l_1 and l_2, that arise from A and B (respectively)(see Figure 7). Note that S_A is the length of $\overline{AD}$ and S_B is the length of $\overline{BE}$. Let $\overline{PQ}$ and $\overline{PR}$ be the orthogonal segments to l_1 and l_2 that arise from P. Now, given that $\triangle ADB$ and $\triangle PRB$ are similar, we have

$$\frac{S_A}{AB} = \frac{PR}{PB}.$$

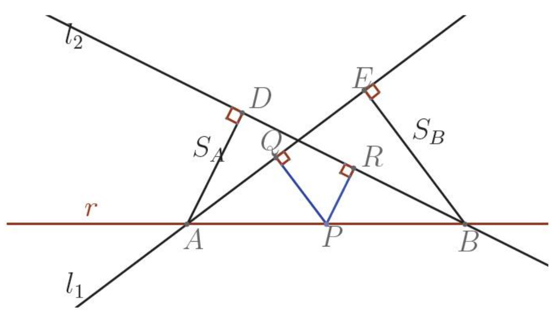

Figure 7: Initial two lines, l_1 and l_2.

$\triangle BEA$ and $\triangle PQA$ are similar too, then

$$\frac{S_B}{AB} = \frac{PQ}{AP}.$$

Let us now check that the rate of change of S_P, with respect to A, is constant:

$$\begin{aligned}\frac{S_A - S_P}{AP} &= \frac{S_A - (PQ + PR)}{AP}\\ &= \frac{S_A}{AP} - \frac{PQ}{AP} - \frac{PR}{AP}\\ &= \frac{S_A}{AP} - \frac{PQ}{AP} - \frac{S_A(PB)}{(AB)(AP)}\\ &= \frac{S_A}{PA} - \frac{S_B(AP)}{(AB)(AP)} - \frac{S_A(PB)}{(AB)(AP)}\\ &= \frac{S_A(AB) - S_B(AP) - S_A(PB)}{(AB)(AP)}\\ &= \frac{S_A(AB - PB) - S_B(AP)}{(AB)(AP)}\\ &= \frac{S_A(AP) - S_B(AP)}{(AB)(AP)}\\ &= \frac{S_A - S_B}{AB}.\end{aligned}$$

Note that S_A, S_B and AB are fixed, then $\frac{S_A - S_B}{AB}$ is a constant, in other words, the quotient $\frac{S_A - S_P}{AP}$ is the same for all P on $\overline{AB}$.

Let us suppose now that the rate of change of S_P with respect to A is constant for all P on $\overline{AB}$ (in the case of m given lines). Set

$$k = \frac{S_P - S_A}{AP} = \frac{S_B - S_A}{AB},$$

where S_A and S_B are the sum of the distances from A and B to the m lines (respectively). If we consider one more line, l, and take l_A and l_B to be the distances from A to l and from B to l, respectively (see Figure 8), we get:

$$\frac{(S_B + l_B) - (S_A + l_A)}{AB} = \frac{S_B - S_A}{AB} + \frac{l_B - l_A}{AB} = k + \frac{l_B - l_A}{AB}.$$

Note that $k + \frac{l_B - l_A}{AB}$ is again a constant. On the other hand, if l_P is the distance from P to the new line, we have:

$$\frac{(S_P + l_P) - (S_A + l_A)}{AP} = \frac{S_P - S_A}{AP} + \frac{l_P - l_A}{AP} = k + \frac{l_P - l_A}{AP}.$$

We have to proof that

$$\frac{l_B - l_A}{AB} = \frac{l_P - l_A}{AP}. \qquad (*)$$

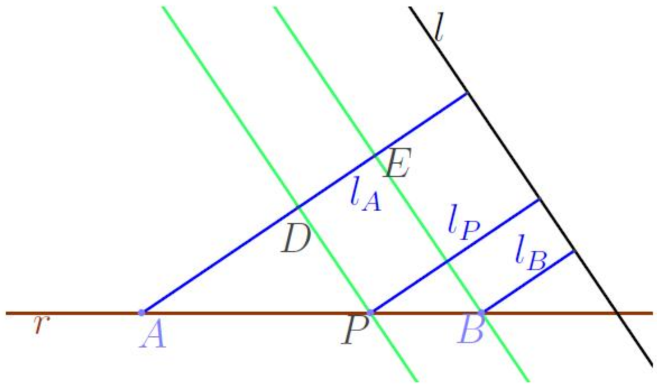

Figure 8: Distances from A, B and P to the added line l.

Now we draw parallel lines to l through P and B respectively (see Figure 8), those lines cut the orthogonal to l from A in D and E respectively. We can say, based on the fundamental theorem of proportionality (see [3]), that $\triangle ADP$ and $\triangle AEB$ are similar and therefore

$$\frac{AD}{AP} = \frac{AE}{AB}.$$

But, $\frac{AD}{AP} = \frac{|l_P - l_A|}{AP}$ and $\frac{AE}{AB} = \frac{|l_B - l_A|}{AB}$, which prove $(*)$. Consequently, the rate of change is constant for all P. □

Now we are ready to prove proposition 1, using lemma 1.

Proof of proposition 1. Based on lemma 1, we say that the graph of S_P when P is on $\overline{AB}$ is a segment; it is easy to see that if P varies along r, S_P is continuous and therefore the graph of S_P is a polygonal line with n vertices corresponding to the intersection of r with the given lines (see Figure 9). Furthermore, we can claim that if P moves away from the extremal intersection points, S_P increases, and thus the polygonal line is open. Consequently, the polygonal line must have at least a minimum on one of its vertices.

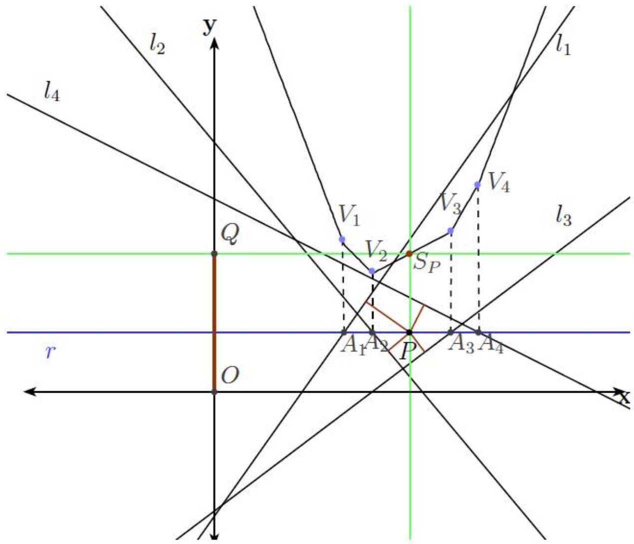

Figure 9: Polygonal line correspond to S_P, with P in r.

To see what happens with S_P as P varies on all the plane, we can vary the auxiliar line r parallely to itself, generating an infinity of polygonal lines, one for each position of r. On each one of these polygonal lines, S_P reaches a minimum value when P is on the intersection of r with some given line, and therefore, if the minimum of S_P on the plane exists, it must be a point M on some of the given lines. Thus, $M \in l_1 \cup l_2 \cup \cdots \cup l_n$, in this way, we can consider the function S_P restricted to the union $l_1 \cup l_2 \cup \cdots \cup l_n$.

Each line l_i intersects at least another given line (because not all lines are parallel). Thus, for l_i we can argue as we have done for r and the graph of S_P. For P restricted to l_i, this graph is a polygonal line up-opened with vertices corresponding to the intersection points of l_i with all the other given lines. This polygonal line contains at least one minimizer M_i, with $M_i \in l_i \cap l_j$, and this is true for all $j = 1, 2, \cdots, n$, $j \neq i$.

Then, we have to examine at most n polygonal lines, each one corresponding to one of the given lines, and therefore the minimum of S_P will be min-$\{S_{M_1}, S_{M_2}, \cdots, S_{M_n}\}$, and it is clearly placed on one of the intersection points M_i. Consequently the function S_P reach its minimum on one of the intersection points of the given lines. □

As we said in the introduction, S_P does not necessarily reach its minimum value at a single point; it could happen that S_P reaches its minimum value at all points on an entire line, on a segment or even on a plane region. We will discuss these cases in a second paper.

4 Conclusions

We solved the problem MIN using CABRI II plus in two ways: to represent instances of the problem and to compute the solution-set of those instances. The software allowed us to consider a huge number of instances, and the huge number of experiments we performed, allowed us to detect useful regularities in the solution-sets we computed. Then, we could propose a suitable conjecture which, with some work, could be formally proved.

Although our solution is not new, we claim that our experimental approach is of interest and could be used for research and education.

References

[1] Borwein, J., et al. Experimentation in mathematics, computational paths to discovery. A. K. Peters. USA, 2004.

[2] Baccaglini-Frank, A., and Mariotti, M.A. Conjecturing and Proving in Dynamic Geometry: the Elaboration of Some Research Hypotheses. In Proceedings of the 6th Conference on European Research in Mathematics Education, Lyon, January 2009

[3] Moise, E., and Downs, F. Jr. Geometría Moderna. Fondo Educativo Interamericano, S.A. Massachusetts, 1964.

[4] Mongomery, D., Peck,E., and Vining, G. Introducción al Análisis de Regresión lineal. Alay Ediciones, S.L. México, 2002.

[5] Barbara, R. The Fermat-Torricelli Points of n Lines. The Mathematical Gazette, Vol. 84, No. 499, pp. 24-29, Mar., 2000

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