

Stationary Boltzmann equation with boundary data depending on the Maxwellian

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Abstract

We obtain the critical points of a functional defined on $L^2(\Omega \times \mathbb{R}^n)$, which coincide with solutions for Boltzmann's stationary equation.

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1 Introduction

Let us consider the following problem: Find $u(x, v) \geq 0$, $u \in L^2(\Omega \times \mathbb{R}^n)$ such that:

$$\begin{cases} v \cdot \nabla_x u = Q(u, u), & \text{a.e } x \in \Omega, v \in B_{3R}(0) \subseteq \mathbb{R}^n; \\ u(x, v) = M(v) = Ae^{-\beta|v|^2}, & x \in \partial\Omega, v \in B_{3R}(0) \subseteq \mathbb{R}^n; \text{ and} \\ \text{exist } x_0 \in \partial\Omega; u(x_0, v) = 0, & v \in B_{3R}(0). \end{cases} \quad (1)$$

Here Q is the non-linear operator of collision given by:

$$Q(u, u) = \int_{B_{3R}(0)} \int_{|w|=1} [w \cdot (v - z)] w [u(z')u(v') - u(z)u(v)] dz dw$$

and Ω is bounded and regular, defined as: For every $x_0 \in \partial\Omega$ there is a neighborhood U of $x_0 \in \mathbb{R}^n$ and a continuously differentiable function $\varphi : U \rightarrow \mathbb{R}$ such that:

- i) $\nabla\varphi(x) \neq 0$ if $x \in U$
- ii) $\partial\Omega \cap U = \{x \in U : \varphi(x) = 0\}$
- iii) $\Omega \cap U = \{x \in U : \varphi(x) < 0\}$;

being

$$\begin{cases} v' = v - [w \cdot (v - z)]w \\ z' = z + [w \cdot (v - z)]w \end{cases} \quad (2)$$

In which (z, v) and (z', v') are speeds of pre-collision and post-collision respectively. We suppose that $z, v \in B_R(0)$ in \mathbb{R}^n , therefore $z', v' \in B_{3R}(0)$ and w is a unit vector. Here we have used the notation $u(z) = u(x, z)$, $u(v) = u(x, v)$, etc, and $[w \cdot (v - z)]w$ is the *kernel* of the operator of collision. $Q : B_R(0) \times B_R(0) \rightarrow \mathbb{R}$ only depends on the speed of the particle.

In physics and application in general, it is important to consider the flow of fluids (gases) with particles that come in and out in the presence of obstacles. These problems are modelled by the addressed problem.

Additional applications of the stationary Boltzmann's equation can be found in traffic flow, calcium metabolism and blood clotting, (see [5], [6], [9], [10], [13]). Problems like this type have been studied in the literature in the setting of L^1 by techniques of weak compactness. So far the variational method has not been used to solve the problems (see [3], [4], [7]). Our group studied problems of fluids by variational methods in which the flow in the boundary is null (see [11],[12]). Here we study this problem by variational methods and we will develop it in seven lemmas and the theorem that solves the problem.

2 Preliminaries

We will study the problem (1) in the Hilbert's space

$$H = \{u \in L^2(\Omega \times \mathbb{R}^n) : \nabla_x u \in L^2(\Omega \times \mathbb{R}^n)\}$$

with

$$\|u\|_H := \|u\|_{L^2(\Omega \times \mathbb{R}^n)}^* + \|\nabla_x u\|_{L^2(\Omega \times \mathbb{R}^n)}^*$$

in which

$$\|u\|_{L^2(\Omega \times \mathbb{R}^n)}^{*2} := \left[\int_{\Omega} \int_{B_{3R}(0)} |v \cdot x| |u(x, v)|^2 dx dv \right]$$

for any $v \in B_R(0) \subseteq \mathbb{R}^n$.

Lemma 1. $\operatorname{div}_x[(v \cdot x)\nabla_x u] = v \cdot \nabla_x u + (v \cdot x)\Delta_x u$, with $u \in H$.

Proof. Obvious. □

Lemma 2. Let $\Omega \subseteq \mathbb{R}^n$, bounded and regular such that there is \hat{n} normal vector to $\partial\Omega$ and suppose $v \cdot \nabla_x u \in L^1(\Omega \times \mathbb{R}^n)$, $\nabla_x u \in L^1(\Omega \times \mathbb{R}^n)$, $\operatorname{div}_x(\cdot) \in L^1(\Omega \times \mathbb{R}^n)$ and $Q(u, u) \in L^1(\Omega \times \mathbb{R}^n)$, then the problem (1) is equivalent to:

$$- \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x)\nabla_x u \, dx dv = \int_{\Omega} \int_{B_{3R}(0)} Q(u, u) \, dx dv$$

a.e. in $\Omega \times B_R(0)$.

Proof. Suppose that we have (1), then integrating, we obtain that

$$\int_{\Omega} \int_{B_{3R}(0)} v \cdot \nabla_x u \, dx dv = \int_{\Omega} \int_{B_{3R}(0)} Q(u, u) \, dx dv$$

applying lemma 1 we obtain:

$$\begin{aligned} \int_{\Omega} \int_{B_{3R}(0)} \operatorname{div}_x [(v \cdot x) \nabla_x u] \, dx dv - \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \Delta_x u \, dx dv \\ = \int_{\Omega} \int_{B_{3R}(0)} Q(u, u) \, dx dv \end{aligned}$$

now applying the divergence theorem we obtain:

$$- \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \Delta_x u \, dx dv = \int_{\Omega} \int_{B_{3R}(0)} Q(u, u) \, dx dv. \quad (3)$$

Now if $v \cdot \nabla_x u \neq Q(u, u)$, if $v \cdot \nabla_x u > Q(u, u)$, $\int_{\Omega} \int_{B_{3R}(0)} [v \cdot \nabla_x u - Q(u, u)] = 0$, it implies that $v \cdot \nabla_x u - Q(u, u) = 0$ a.e. which is a contradiction. Same if $v \cdot \nabla_x u < Q(u, u)$. From now, we will focus on studie of the problem (2). \square

Lemma 3. *Let G a convex function and Frechet differentiable function in $u \in L^2(\Omega \times B_{3R}(0))$ and let*

$$J(u) := \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 \, dx dv - G(u)$$

with

$$G'(u)p = \int_{\Omega} \int_{B_{3R}(0)} p(x, v) Q(u, u) \, dx dv \forall p \in H \text{ and } G(0) = 0$$

then

$$J'(u)p = - \int_{\Omega} \int_{B_{3R}(0)} p(x, v) [(v \cdot x) \nabla_x u + Q(u, u)] \, dx dv, \text{ for all } p \in H.$$

Proof.

$$\begin{aligned}
J'(u)p &= \lim_{h \rightarrow 0} \frac{1}{h} [J(u + hp) - J(u)] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x(u + hp)\|^2 dx dv \right. \\
&\quad \left. - G(u + hp) - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv + G(u) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|(\nabla_x u + h \nabla_x p)\|^2 dx dv \right. \\
&\quad \left. - G(u + hp) - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv + G(u) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv + \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) h \nabla_x u \cdot \nabla_x p dx dv \right. \\
&\quad \left. - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv \right] - G'(u)p \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) h \nabla_x u \cdot \nabla_x p dx dv \right] - G'(u)p \\
&= - \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) p(x, v) \Delta_x u dx dv - G'(u)p.
\end{aligned}$$

As $(u(x, v) = M(u) = Ae^{-\beta|v|^2}$, $x \in \partial\Omega$ and $v \in \mathbb{R}^n$, such that $\nabla_x u = 0$, $x \in \partial\Omega$, $v \in \mathbb{R}^n$; $u(x, v) = cte$ with $x \in \partial\Omega$, $v \in \mathbb{R}^n$, but as it exists $x_0 \in \partial\Omega$: $u(x_0, v) = 0$, it implies that $u = 0$ in $\partial\Omega$, therefore

$$J'(u)p = \int_{\Omega} \int_{B_{3R}(0)} p(x, v)(v \cdot x) \Delta_x u dx dv - \int_{\Omega} \int_{B_{3R}(0)} p(x, v) Q(u, u) dx dv$$

□

As a consequence of the former lemma, finding critical points of J is equivalent to find solutions of Boltzmann's stationary equation.

Lemma 4. *If $(v \cdot x) \neq 0$ and $\int_{\Omega} \int_{B_{3R}(0)} \frac{\|v - z\|}{|v \cdot x|} dv dp \geq 0$, then, it exists $k \geq 0$ (real) such that $\|Q(u, u)\|_{L^1(\Omega \times B_{3R}(0))} \leq k \|u\|_{L^2(\Omega \times B_{3R}(0))}^2$. (If $(v \cdot x) = 0$, then by lemma 1, we have that $v \cdot \nabla_x u = 0$, and if u is solution of (1), then $Q(u, u) = 0$ and the inequality result is trivial).*

Proof. By definition

$$\begin{aligned}
|Q(u, u)| &\leq \int_{B_{3R}(0)} \int_{|p|=1} \|u - z\| |u(x, z')u(x, v') - u(x, z)u(x, v)| dz dp \\
&\leq \int_{B_{3R}(0)} \int_{|p|=1} \|u - z\| |u(x, z')u(x, v') + u(x, z)u(x, v)| dz dp \quad (u \geq 0) \\
&\leq \int_{B_{3R}(0)} \int_{|p|=1} \|u - z\| |u(x, z')u(x, v')| dz dp \\
&\quad + \int_{B_{3R}(0)} \int_{|p|=1} \|u - z\| |u(x, z)u(x, v)| dz dp \\
&\leq \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} [u^2(x, z') + u^2(x, v')] dz dp \\
&\quad + \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} [u^2(x, z) + u^2(x, v)] dz dp
\end{aligned}$$

then

$$\begin{aligned}
&\int_{\Omega} \int_{B_{3R}(0)} |Q(u, u)| dx dv \\
&\leq \int_{\Omega} \int_{B_{3R}(0)} \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} u^2(x, z') dz dp dx dv \\
&\quad + \int_{\Omega} \int_{B_{3R}(0)} \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} u^2(x, v') dz dp dx dv \\
&\quad + \int_{\Omega} \int_{B_{3R}(0)} \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} u^2(x, z) dz dp dx dv \\
&\quad + \int_{\Omega} \int_{B_{3R}(0)} \int_{B_{3R}(0)} \int_{|p|=1} |v \cdot x| \frac{1}{2} \frac{\|u - z\|}{|v \cdot x|} u^2(x, v) dz dp dx dv
\end{aligned}$$

i.e.

$$\|Q(u, u)\|_{L^1(\Omega \times B_{3R}(0))} \leq 2 \int_{B_{3R}(0)} \int_{|p|=1} \frac{\|u - z\|}{|v \cdot x|} dz dp \|u\|_{L^2(\Omega \times B_{3R}(0))}^2$$

defining

$$k = 2 \int_{B_{3R}(0)} \int_{|p|=1} \frac{\|u - z\|}{|v \cdot x|} dz dp$$

then

$$\|Q(u, u)\|_{L^1(\Omega \times B_{3R}(0))} \leq k \|u\|_{L^2(\Omega \times B_{3R}(0))}^2.$$

□

Remark 1. If $\omega \in [0, u] \in B^*(0, R^*) \subseteq L^2(\Omega \times B_{3R}(0))$, being $[0, u]$ line segment, then

$$\|Q(\omega, \omega)\|_{L^1(\Omega \times B_{3R}(0))} \leq k(R^*)^2 \forall \omega \in [0, u].$$

Lemma 5. J is coercive.

Proof. As G is Gateaux differentiable, then by the mean value theorem

$$\begin{aligned} |G(u) - G(0)| &\leq \|u - 0\| \text{Sup}_{\omega \in [0, u]} \int_{B_{3R}(0)} \int_{\Omega} |Q(\omega, \omega)| dx dv \\ &\leq \|u\|_{L^2(\Omega \times B_{3R}(0))} \text{Sup}_{\omega \in [0, u]} \|Q(\omega, \omega)\|_{L^1(\Omega \times B_{3R}(0))} \end{aligned}$$

then $|G(u)| \leq k(R^*)^2 \|u\|_{L^2(\Omega \times B_{3R}(0))}$, $-k(R^*)^2 \|u\|_{L^2(\Omega \times B_{3R}(0))} \leq -G(u)$, i.e,

$$\begin{aligned} J(u) &> \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv - k(R^*)^2 \|u\|_{L^2(\Omega \times B_{3R}(0))} \\ &= \frac{1}{2} \|u\|^2 - k(R^*)^2 \|u\|, \end{aligned}$$

then if $\|u\| \rightarrow \infty$, then $J(u) \rightarrow \infty$, J is coercive. \square

Lemma 6. J is w.l.s.c.

Proof. $u \rightarrow \|u\|^2$ is w.l.s.c. and as G is convex and strongly continuous, then J is w.l.s.c. \square

Lemma 7. J is differentiable in $u \in H$.

Proof. Let $\delta J \equiv J(u + h) - J(u)$. See that

$$\begin{aligned}
\delta J &= \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x(u+h)\|^2 dx dv - G(u+h) \\
&\quad - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv + G(u) \\
&= \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u + \nabla_x h\|^2 dx dv - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv \\
&\quad - G(u+h) + G(u) \\
&= \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv + \frac{2}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \nabla_x u \cdot \nabla_x h dx dv \\
&\quad + \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x h\|^2 dx dv - \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x u\|^2 dx dv \\
&\quad - [G(u+h) - G(u)] \\
&= \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \nabla_x u \cdot \nabla_x h dx dv + \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x h\|^2 dx dv \\
&\quad - [G(u+h) - G(u)] \\
&= - \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) h \Delta_x u dx dv - [G(u+h) - G(u)] \\
&\quad + \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x h\|^2 dx dv \\
&= - \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) h \Delta_x u dx dv - DG(u)h + o(h) \\
&\quad + \frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x h\|^2 dx dv.
\end{aligned}$$

Now

$$\lim_{h \rightarrow 0} \left[\frac{\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) \|\nabla_x h\|^2 dx dv}{\|h\|} \right] = \lim_{h \rightarrow 0} \left[\frac{\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) (\nabla_x h \cdot \nabla_x h) dx dv}{\sqrt{(h, h)}} \right]$$

When $h \rightarrow 0$ it implies that $\frac{\frac{1}{2} \int_{\Omega} \int_{B_{3R}(0)} (v \cdot x) (\nabla_x h \cdot \nabla_x h) dx dv}{\|h\|} \rightarrow 0$, therefore it is concluded that J is differentiable in u . \square

Theorem 1. *We conclude of the previous hypotheses on Ω , G , Q and*

$$\int_{\Omega} \int_{B_{3R}(0)} \frac{\|v \cdot z\|}{|v \cdot x|} dv dp \geq 0 \text{ if } (v \cdot x) \neq 0,$$

that the problem (1) has at least one solution.

Proof. Applying Theorem 5.5 of [1], it implies that J has a global minimum, i.e. there is $z \in H$, such that $J(z) = \min\{J(u) : u \in L^2(\Omega \times \mathbb{R}^n)\}$, as J is differentiable at z , then $J'(z) = 0$, i.e. z is a critical point of J and also z is a solution of (2). \square

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