Vol. XX, N$^{\underline{o}}$ 2, Diciembre (2012)
Matemáticas: 3–23

Matemáticas:
Enseñanza Universitaria

Solitary wave solutions of a weakly dispersive system

Juan Carlos Muñoz Grajales
Universidad del Valle

Ivonne Rivas Triviño
Instituto Nacional de Matemática Pura e Aplicada, IMPA

Received Oct. 04, 2012 Accepted Nov. 13, 2012

Abstract

We establish a theorem on global well-posedness of an initial value problem associated to a generalized Boussinesq-type system, extending the results presented by Muñoz and Rivas in Muñoz (2008). The Hamiltonian structure of this system is described, and we further show that it has solitary wave solutions for a certain range of wave velocity by using the theory implemented in Toland (1986).

Keywords: Cauchy problem, global solutions, solitary wave solutions.

MSC(2000): 42A45, 47A75, 35Q51, 76M45

Resumen

En este trabajo establecemos un teorema de buena colocación global de un problema de valor inicial asociado a un sistema de tipo Boussinesq generalizado, extendiendo los resultados presentados por Muñoz and Rivas en Muñoz (2008). Se describe la estructura Hamiltoniana de este sistema, y además demostramos que posee soluciones de onda solitaria para cierto rango de velocidad de onda usando la teoría desarrollada en Toland (1986).

Palabras y frases claves: Problema de Cauchy, soluciones globales, soluciones de onda solitaria.

1 Introduction

In this paper we consider the dispersive system

$$\begin{cases} \eta_t + \delta\eta_{xxt} + \mu u_{xxt} + u_x + \theta u_{xxx} + \alpha(u^p\eta)_x = 0, \quad (x,t) \in \mathbb{R}\times[0,\infty), \\ u_t + \delta u_{xxt} + \mu\eta_{xxt} + \eta_x + \theta\eta_{xxx} + \alpha\Big(\dfrac{u^{p+1}}{p+1}\Big)_x = 0, \\ u(x,0) = u_0(x), \quad \eta(x,0) = \eta_0(x), \end{cases} \tag{1}$$

where δ, μ, α and p are modelling parameters such as $\delta < \mu \leq 0$, θ, $\alpha \in \mathbb{R}$ and $p \geq 1$. For $p = 1$, $\mu = 0$, system (1) corresponds directly to a physical model derived by Bona et al. [2] from the full Euler equations for modelling the motion of small-amplitude long waves on the surface of a flat bottom channel which contains an ideal fluid under the force of gravity. The function $u = u(x,t)$ represents the velocity of the flow and $\eta = \eta(x,t)$ denotes the height of the wave with respect to an equilibrium level. The Boussinesq-type systems such as (1) are reduced models which are very important in both engineering applications as well as in laboratory scales, because in general the full Euler equations appear more complicated than necessary. The original Boussinesq equations applicable for waves in channels with flat bottom are due to Boussinesq [5], and then Peregrine [16] derived the first set of equations for waves propagating over a channel with variable depth but satisfying the mild-slope hypothesis. Later, Nwogu [15], Schäffer and Madsen [17], Madsen et al. [10], [11] among others, introduced improved Boussinesq-type equations corresponding to Padé expansions of the Stokes linear dispersion relation for waves of arbitrary depth. Very recently, Muñoz and Nachbin [12], [14] derived new forms of Boussinesq models applicable even for waves propagating over a channel with highly variable topography in a certain wave regime. Thus, these models apply to a wider regime of topographies than the previous ones. They also explored the accuracy of the Boussinesq-type approximations with regard to the full Euler equations. This amount of research concerning Boussinesq formulations and additional modelling issues motivate our study of the properties of solutions to equations (1) with nonlinear terms with exponent $p \geq 1$ because we are interested in studying the effect of generalized nonlinearities on the solutions of this dispersive system. Moreover the terms with the parameter $\mu < 0$ incorporate additional dispersion effects in equations (1).

Under some regularity conditions on the initial data, Bona et al. [3] showed that there exists a unique global solution pair (u, η) of system (1) satisfying $(u, \eta) \in (C(\mathbb{R}_+; H^s(\mathbb{R})))^2$, $s \geq 1$, but only in the case $p = 1$, $\mu = 0$ and $\theta < 0$. In the present paper, we extend this result to consider more general values of the modelling parameters in system (1). More precisely, we prove that there exists a unique solution pair $(u, \eta) \in (C([0, T]; H^s(\mathbb{R})))^2$ for each $T > 0$ of the Cauchy problem (1) and (u_t, η_t) belongs to $(C([0, T]; H^{s-1}(\mathbb{R})))^2$, provided the initial data and parameters satisfy that $u_0, \eta_0 \in H^s(\mathbb{R})$, $s \geq 1$ integer, $\delta < \mu < 0$, $0 < \mu - \delta < 1$, $1 < -\delta - \mu$, $\theta, \alpha \in \mathbb{R}$ and $p \geq 1$. This global well-posedness result for system (1) improves upon the one developed by Muñoz and Rivas in [13].

In second place, if $\delta < \mu \leq 0$, $\theta > 0$, $\alpha > 0$, $p \geq 1$, we prove that system (1) has solitary wave solutions

$$u(x, t) = u(\xi), \qquad \eta(x, t) = \eta(\xi), \qquad \xi = x - ct,$$

where $\eta(\xi)$, $u(\xi)$ and all their derivatives tend to zero as $|\xi| \to \infty$, provided that the wave speed c satisfies $c > \max\{1, \frac{\theta}{\mu - \delta}\}$. Observe that the functions $\eta(\xi), u(\xi)$ must satisfy the system

$$\begin{aligned} &(\mu c - \theta)u'' + c\delta\eta'' - u + c\eta - \alpha(u^p\eta) = 0, \\ &c\delta u'' + (\mu c - \theta)\eta'' + cu - \eta - \alpha\left(\frac{u^{p+1}}{p+1}\right) = 0. \end{aligned} \tag{2}$$

We also compute approximations to these solitary wave solutions in the case when the model's parameters are small.

The solitary wave solutions have played an important role in a broad set of applications such as, Fluid Dynamics, Optics, Acoustics, Oceanography and

Weather Forecasting, among others. A recent spectacular application is the use of optical solitons in fibers as an efficient (reliable and fast) technique of long-distance communication (see the work by Kivshar and Agrawal [9]). Thus, this class of solutions and their properties such as existence, orbital stability/instability under small perturbations and other related issues are of great interest and have captured the attention of researchers who used both analytical and numerical means.

Due to the decay properties at infinity of a solitary wave solution, it is clear that a homoclinic orbit about the origin $(0,0)$ of equations (2) leads to a solitary wave solution of system (1). To study these solutions, we adapt and extend the techniques developed by Chen in [8] for system (1) with $p=1$, and $\mu=0$. The main result used in his proof is an interesting theorem due to Toland [18] which provides necessary conditions for a system of two second-order ordinary differential equations in the form of equations (2) to admit homoclinic solutions. Other papers related to homoclinic solutions are those by Champneys et al. [6], [7] where a very useful numerical toolbox for analyzing the bifurcation of homoclinic orbits in reversible systems was introduced. Moreover, Amick and Toland [1] developed a result on uniqueness of homoclinic orbits for a class of fourth-order equations.

The scheme of the paper is as follows. Section 2 contains the definition of the functional spaces and notation required for studying existence of solutions of equations (1). Furthermore by using the Fourier transform, we rewrite it as a system of integral equations in a convenient way, in order to apply the Banach fixed point theorem. In Section 3, we establish the global existence of a solution of equations (1). In Section 4, the Hamiltonian structure for system (1) is described. In Section 5 we discuss the existence of solitary wave solutions of the model within certain range of wave velocity and model's parameters. In section 6 we discuss how to approximate solitary wave solutions of system (1). Finally, in Section 7 we give the conclusions of the work.

2 Problem setting and notation

We will use the standard notation. For $1 \le p \le \infty$ we will denote by $L^p(\mathbb{R})$ the Banach space of measurable functions in $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|^p dx < \infty$ if $1 \le p < \infty$ or ess $\sup_{\mathbb{R}}|f| < \infty$ if $p=\infty$. We define the norm in $L^p(\mathbb{R})$ for $1 \le p < \infty$ by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p},$$

and in $L^\infty(\mathbb{R})$ by $\|f\|_\infty = \text{ess sup}_{\mathbb{R}} |f|$. To simplify the notation we set $\|f\| = \|f\|_{L^2}$ and $\|f\|_p = \|f\|_{L^p}$. For a function f in L^1 the Fourier transform is defined as $\mathfrak{F}(f)(y) = \hat{f}(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x)dx,\ y \in \mathbb{R}$ and the inverse Fourier transform $\mathfrak{F}^{-1}(f)(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} f(x)dx$. These two operators can be extended to

$L^2(\mathbb{R})$. Let us define the convolution of two functions f, $g \in L^2$ as

$$f * g(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Thus $\widehat{f * g} = \hat{f}\hat{g}$.

For $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R})$ as the completion of the Schwartz space $S(\mathbb{R})$ with respect to the norm

$$\|f\|_{H^s} = \Big(\int_{\mathbb{R}} (1+y^2)^s |\hat{f}(y)|^2 dy \Big)^{1/2}.$$

The product norm is defined by $\|(u,\eta)\|_{H^s \times H^s} = \|u\|_{H^s} + \|\eta\|_{H^s}$, for $(u,\eta) \in H^s \times H^s$. We denote by $C([0,T]; H^s)$ the space of continuous functions $f : [0,T] \to H^s$. We use the notation $H^s_T \equiv C([0,T]; H^s)$, i.e. the space of continuous functions $t \to f(t,.) \in H^s$, $t \in [0,T]$, with the norm $\|f\|_{H^s_T} = \sup_{t \in [0,T]} \|f(t,.)\|_{H^s}$ and the product norm defined by $\|(u,\eta)\|_{H^s_T \times H^s_T} = \|u\|_{H^s_T} + \|\eta\|_{H^s_T}$, for $(u,\eta) \in (H^s_T)^2 := H^s_T \times H^s_T$.

Suppose that the model's parameters satisfy $\delta < \mu \leq 0$, θ, $\alpha \in \mathbb{R}$, $p \geq 1$. In the first place, we rewrite the Cauchy problem (1), by using the Fourier transform with respect to x as follows:

$$\alpha \widehat{U_t} + iy\mathbf{B}\widehat{U} + \widehat{F(U)} = 0, \tag{3}$$

where

$$\alpha = \begin{pmatrix} -\mu y^2 & 1-\delta y^2 \\ 1-\delta y^2 & -\mu y^2 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1-\theta y^2 & 0 \\ 0 & 1-\theta y^2 \end{pmatrix},$$

$$F(U) = \begin{pmatrix} F_{1x}(u,\eta) \\ F_{2x}(u,\eta) \end{pmatrix},$$

$$F_1(u,\eta) = \alpha u^p \eta, \quad F_2(u,\eta) = \alpha \frac{u^{p+1}}{p+1},$$

and

$$U = \begin{pmatrix} u \\ \eta \end{pmatrix}.$$

This is the same strategy as used in [13]. Then taking the Fourier transform in equation (3) yields

$$\widehat{U}_t = -iy\alpha^{-1}\mathbf{B}\widehat{U} - \alpha^{-1}\widehat{F(U)}.$$

We remark that the matrix $\alpha = \alpha(y)$ is non-singular for all frequencies $y \in \mathbb{R}$, since the model's parameters in equations (1) satisfy $\delta < \mu \leq 0$. Therefore

$$\widehat{U}(y,t) = e^{-iy\alpha^{-1}\mathbf{B}t} \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} - \int_0^t e^{-iy\alpha^{-1}\mathbf{B}(t-s)} \alpha^{-1} \widehat{F(U)}(y,s)ds.$$

Taking the inverse Fourier transform, we obtain that

$$U(x,t)=\mathfrak{F}^{-1}\Big(e^{-iy\alpha^{-1}\mathbf{B}t}\mathfrak{F}\begin{pmatrix}u_0\\ \eta_0\end{pmatrix}\Big)-\int_0^t \mathfrak{F}^{-1}\Big(e^{-iy\alpha^{-1}\mathbf{B}(t-s)}\alpha^{-1}\widehat{F(U)}\Big)(x,s)ds.$$

Therefore

$$U(x,t)=S(t)\begin{pmatrix}u_0\\ \eta_0\end{pmatrix}-\int_0^t S(t-s)\mathfrak{F}^{-1}(\alpha^{-1}\widehat{F(U)})(x,s)ds, \tag{4}$$

where

$$S(t)\begin{pmatrix}u\\ \eta\end{pmatrix}=\mathfrak{F}^{-1}\Big(e^{-iy\alpha^{-1}\mathbf{B}t}\mathfrak{F}\begin{pmatrix}u\\ \eta\end{pmatrix}\Big), \quad t\geq 0. \tag{5}$$

Observe that

$$\alpha^{-1}\widehat{F(U)}=\begin{pmatrix}k_1 & k_2\\ k_2 & k_1\end{pmatrix}\begin{pmatrix}\widehat{F_{1x}}\\ \widehat{F_{2x}}\end{pmatrix}=\begin{pmatrix}\widehat{K}_1 & \widehat{K}_2\\ \widehat{K}_2 & \widehat{K}_1\end{pmatrix}\begin{pmatrix}\widehat{F_{1x}}\\ \widehat{F_{2x}}\end{pmatrix},$$

with

$$k_1=\widehat{K}_1=\frac{-\mu y^2}{(\mu^2-\delta^2)y^4+2\delta y^2-1}, \quad k_2=\widehat{K}_2=\frac{\delta y^2-1}{(\mu^2-\delta^2)y^4+2\delta y^2-1},$$

which implies that

$$\mathfrak{F}^{-1}\Big(\alpha^{-1}\widehat{F(U)}\Big)=\mathfrak{F}^{-1}\begin{pmatrix}\widehat{K}_1\widehat{F_{1x}}+\widehat{K}_2\widehat{F_{2x}}\\ \widehat{K}_2\widehat{F_{1x}}+\widehat{K}_1\widehat{F_{2x}}\end{pmatrix}=\begin{pmatrix}K_1*F_{1x}+K_2*F_{2x}\\ K_2*F_{1x}+K_1*F_{2x}\end{pmatrix}.$$

Thus equation (4) can also be written as

$$U(x,t)=S(t)\begin{pmatrix}u_0\\ \eta_0\end{pmatrix}+\int_0^t S(t-s)G(U(x,s))ds, \tag{6}$$

where

$$G(U)=-\begin{pmatrix}K_1*F_{1x}+K_2*F_{2x}\\ K_2*F_{1x}+K_1*F_{2x}\end{pmatrix}.$$

On the other hand, note that the Boussinesq system (1) can also be written as

$$U_t=\mathcal{A}^{-1}\mathcal{B}U-\mathcal{A}^{-1}(F(U)), \tag{7}$$

where $\mathcal{A},\mathcal{B}$ are the linear differential operators

$$\mathcal{A}:=\begin{pmatrix}\mu\partial_x^2 & I+\delta\partial_x^2\\ I+\delta\partial_x^2 & \mu\delta_x^2\end{pmatrix}, \quad \mathcal{B}:=\begin{pmatrix}-\partial_x-\theta\partial_x^3 & 0\\ 0 & -\partial_x-\theta\partial_x^3\end{pmatrix}$$

and I denotes the identity operator. We remark that the operators $\mathcal{A}:H^2\times H^2\to L^2\times L^2$, $\mathcal{A}^{-1}:L^2\times L^2\to H^2\times H^2$ and $\mathcal{B}:H^3\times H^3\to L^2\times L^2$ can be characterized by using the Fourier transform as

$$\mathcal{A}f=\mathfrak{F}^{-1}(\alpha(y)\hat{f}(y)), \quad \mathcal{A}^{-1}f=\mathfrak{F}^{-1}(\alpha^{-1}(y)\hat{f}(y)), \quad \mathcal{B}f=\mathfrak{F}^{-1}(-iy\mathbf{B}\hat{f}(y)),$$

and $G(U)=-\mathcal{A}^{-1}F(U)$.

3 Global existence and uniqueness in $H^s(\mathbb{R})$

In [13] the authors proved the following theorem on the existence of local solutions of the integral equation (6).

Theorem 1. *Let $u_0, \eta_0 \in H^s(\mathbb{R}), s \geq 1,\ \delta < \mu \leq 0,\ \theta, \alpha \in \mathbb{R}$ and $p \geq 1$. Then there exists a positive constant $T_0 > 0$ and a unique solution pair $(u, \eta) \in (C([0, T_0]; H^s(\mathbb{R})))^2$ of the integral equation (6).*

The following theorem extends the results in [13] establishing that a solution of the integral equation (6) is indeed a solution of the Cauchy problem (1).

Theorem 2. *Suppose the same hypothesis of theorem 1. Then the function $U = (u, \eta)^T$ defined by equation (6) is a solution of the Cauchy problem (1), with the time derivative given by*

$$lim_{h\to 0^+} \left\| \frac{U(t+h) - U(t)}{h} - \mathcal{A}^{-1}\mathcal{B}U(t) + \mathcal{A}^{-1}F(U(t)) \right\|_{H^{s-1}\times H^{s-1}} = 0, \qquad (8)$$

for $t \geq 0$.

Proof. To simplify our notation let $\|.\|_{s-1} = \|.\|_{H^{s-1}\times H^{s-1}}$ and let us denote by $\|.\| = \|.\|_2$ the Euclidean norm in $\mathbb{R}^2$ or the corresponding induced matrix norm. Taking $t \geq 0$ and $h > 0$, we obtain that

$$\left\| \frac{U(t+h) - U(t)}{h} - \mathcal{A}^{-1}\mathcal{B}U(t) + \mathcal{A}^{-1}F(U(t)) \right\|_{s-1} \leq E + M + J,$$

where

$$E := \left\| \frac{1}{h}(S(t+h) - S(t)) \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} - \mathcal{A}^{-1}\mathcal{B}S(t) \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \right\|_{s-1},$$

$$M := \Bigg\| \frac{1}{h} \int_0^t (S(t+h-s) - S(t-s))G(U(x,s))ds$$
$$-\mathcal{A}^{-1}\mathcal{B} \int_0^t S(t-s)G(U(x,s))ds \Bigg\|_{s-1},$$

and

$$J := \left\| \frac{1}{h} \int_t^{t+h} S(t+h-s)G(U(x,s))ds + \mathcal{A}^{-1}F(U(t)) \right\|_{s-1}.$$

Note that since the integral in the quantity J is the mean value of a continuous function over an interval that shrinks to zero as $h \to 0^+$, it must converge in $H^{s-1} \times H^{s-1}$ to the value of the integrand at t, i.e.,

$$\lim_{h\to 0^+} \frac{1}{h} \int_t^{t+h} S(t+h-s)G(U(x,s))ds = -\mathcal{A}^{-1}F(U(t)).$$

As a consequence, J must converge to zero when $h \to 0^+$. We recall that $G(U) = -\mathcal{A}^{-1}F(U)$. On the other hand, the terms E and M tend to zero when $h \to 0^+$. In fact, we have that

$$
\begin{aligned}
&\left\| \frac{1}{h}(S(t+h) - S(t)) \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} - \mathcal{A}^{-1}\mathcal{B}S(t) \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \right\|_{s-1} = \\
&\quad \Bigg\| \mathfrak{F}^{-1}\Big(\frac{1}{h}\Big(e^{-iy\alpha^{-1}\mathbf{B}(t+h)} - e^{-iy\alpha^{-1}\mathbf{B}t}\Big) \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \Big) \\
&\quad -\mathcal{A}^{-1}\mathcal{B}\mathfrak{F}^{-1}\Big(e^{-iy\alpha^{-1}\mathbf{B}t} \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \Big) \Bigg\|_{s-1} = \\
&\quad \int_{\mathbb{R}} (1+y^2)^{s-1} \Bigg\| e^{-iy\alpha^{-1}\mathbf{B}t} \frac{1}{h}\Big(e^{-iy\alpha^{-1}\mathbf{B}h} - I\Big) \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \\
&\quad -(-iy)\alpha^{-1}\mathbf{B} e^{-iy\alpha^{-1}\mathbf{B}t} \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \Bigg\|^2 dy = \\
&\quad \int_{\mathbb{R}} (1+y^2)^{s-1} \Bigg\| e^{-iy\alpha^{-1}\mathbf{B}t} \Big(\frac{1}{h}(e^{-iy\alpha^{-1}\mathbf{B}h} - I) - (-iy)\alpha^{-1}\mathbf{B}\Big) \mathfrak{F} \begin{pmatrix} u_0 \\ \eta_0 \end{pmatrix} \Bigg\|^2 dy.
\end{aligned}
$$

Since the eigenvalues $i\lambda_1, i\lambda_2$ of the matrix $-iy\alpha^{-1}\mathbf{B}$ are imaginary, we have that for any $t \geq 0$, $y \in \mathbb{R}$,

$$\|e^{-iy\alpha^{-1}\mathbf{B}t}\| \leq C,$$

for some constant $C > 0$. Furthermore, by applying the mean value theorem we obtain

$$\left\| \frac{e^{-iy\alpha^{-1}\mathbf{B}h} - I}{h} \right\| \leq C\| - iy\alpha^{-1}\mathbf{B}\|,$$

for all $y \in \mathbb{R}$ and

$$lim_{h\to 0^+} \left\| \frac{e^{-iy\alpha^{-1}\mathbf{B}h} - I}{h} - (-iy)\alpha^{-1}\mathbf{B} \right\| = 0.$$

□

In the following result we extend the local solutions claimed in the previous theorem.

Theorem 3. *Let $u_0, \eta_0 \in H^s(\mathbb{R}), s \geq 1$ integer, $\delta < \mu < 0$, $0 < \mu - \delta < 1$, $1 < -\delta - \mu$ and $\theta, \alpha \in \mathbb{R}$. Furthermore, $\theta, \alpha \in \mathbb{R}$ and $p \geq 1$. Then there exists a unique solution pair $U = (u, \eta) \in (C([0,T]; H^s(\mathbb{R})))^2$ for each $T > 0$ of the Cauchy problem (1) (in the sense given in (8)) and (u_t, η_t) belongs to $(C([0,T]; H^{s-1}(\mathbb{R})))^2$.*

Proof. Let $(u, \eta) \in (C([0,T_0]; H^s(\mathbb{R})))^2$ (with $s \geq 1$ and $T_0 > 0$) be the local solution of equation (6) guaranteed by Theorem 1. Then by Theorem 2 it follows that the time derivatives of u and η exist (in the sense given in (8)), and (u,η) satisfies the Cauchy problem (1).

Observe that the operators $\mathcal{A},\mathcal{B}$ are such that $\mathcal{A}^{-1}: H^{s-3}\times H^{s-3}\to H^{s-1}\times H^{s-1}$, $\mathcal{B}: H^s\times H^s\to H^{s-3}\times H^{s-3}$. Therefore, since $(u(.,t),\eta(.,t))\in H^s\times H^s$ and $(u^p\eta)_x$, $(1/(p+1))u^{p+1})_x\in H^{s-1}$, we conclude from equation (7) that $(u_t,\eta_t)\in (C([0,T_0];H^{s-1}(\mathbb{R})))^2$.

In order to extend this solution for any time $t>0$, let us multiply the first equation in system (1) by η, and the second one by u and integrate with respect to x in $\mathbb{R}$ to obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\eta^2-\delta\eta_x^2)dx+\int_{\mathbb{R}}(\mu u_{xxt}\eta+u_x\eta+\theta\eta u_{xxx}+\alpha(u^p\eta)_x\eta)dx=0,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2-\delta u_x^2)dx+\int_{\mathbb{R}}(\mu\eta_{xxt}u+\eta_x u+\theta\eta_{xxx}u+\alpha u^{p+1}u_x)dx=0.$$

Summing the equations above, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\eta^2+u^2-\delta\eta_x^2-\delta u_x^2-2\mu\eta_x u_x)dx=\alpha\int_{\mathbb{R}}(u^p\eta\eta_x)dx. \tag{9}$$

In this step we used integration by parts, the decay properties of the pair solution $x\to u(x,t),\eta(x,t)$ at $x=\pm\infty$ and the following relationships:

$$\int_{\mathbb{R}}\eta u_{xxx}dx=-\int_{\mathbb{R}}\eta_{xxx}udx$$

$$\int_{\mathbb{R}}(u_{xxt}\eta+\eta_{xxt}u)dx=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(-2\eta_x u_x)dx,$$

$$\int_{\mathbb{R}}(u^p\eta)_x\eta=-\int_{\mathbb{R}}u^p\eta\eta_x dx,$$

and

$$\int_{\mathbb{R}}u^{p+1}u_x dx=\int_{\mathbb{R}}(\frac{u^{p+2}}{p+2})_x dx=0.$$

Using the Hölder inequality in equation (9), we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\eta^2+u^2-\delta u_x^2-2\mu\eta_x u_x)dx&\le|\alpha|\|u^p\|_\infty\|\eta\|\|\eta_x\|\le C(\|\eta\|^2+\|\eta_x\|^2)\\&\le C(\|\eta\|^2+\|\eta_x\|^2+\|u\|^2+\|u_x\|^2).\end{aligned}$$

Hereafter C denotes a generic positive constant independent of time t. Note that since for any fixed $t\in[0,T_0]$ the functions $u(.,t)$, $\eta(.,t)$ belong to $H^s(\mathbb{R})$, $s\ge 1$, we have that they also belong to $L^\infty(\mathbb{R})$ by virtue of the imbedding $H^s\hookrightarrow L^\infty$ for any $s\ge 1$.

Integrating both sides of the last inequality with respect to t and using the fundamental theorem of Calculus in the interval $[0,t]$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}(\eta^2+u^2-\delta\eta_x^2-\delta u_x^2)dx &\le \int_{\mathbb{R}}(\eta_0^2+u_0^2-\delta\eta_{0,x}^2-\delta u_{0,x}^2-2\mu\eta_{0,x}u_{0,x})dx\\ &\quad+2\mu\int_{\mathbb{R}}\eta_x u_x dx+C\int_0^t\int_{\mathbb{R}}(\eta^2+\eta_x^2+u^2+u_x^2)dxds\\ &\le \int_{\mathbb{R}}(\eta_0^2+u_0^2-\delta\eta_{0,x}^2-\delta u_{0,x}^2-\mu\eta_{0,x}^2-\mu u_{0,x}^2)dx\\ &\quad-\int_{\mathbb{R}}(\mu\eta_x^2+\mu u_x^2)dx+C\int_0^t\int_{\mathbb{R}}(\eta^2+\eta_x^2+u^2+u_x^2)dxds.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}}(\eta^2+u^2+(\mu-\delta)\eta_x^2+(\mu-\delta)u_x^2)dx\\ &\qquad\le \int_{\mathbb{R}}(\eta_0^2+u_0^2-(\delta+\mu)\eta_{0,x}^2-(\delta+\mu)u_{0,x}^2)dx\\ &\qquad\quad+C\int_0^t\int_{\mathbb{R}}(\eta^2+\eta_x^2+u^2+u_x^2)dxds.\end{aligned}$$

Since $0<\mu-\delta<1,\ \ 1<-\delta-\mu$ and $\delta<\mu<0$, it is clear that

$$\begin{aligned}&\int_{\mathbb{R}}\Big(\frac{\eta^2}{\mu-\delta}+\frac{u^2}{\mu-\delta}+\eta_x^2+u_x^2\Big)dx\\ &\quad\le -\frac{\delta+\mu}{\mu-\delta}\int_{\mathbb{R}}\Big(-\frac{\eta_0^2}{\delta+\mu}-\frac{u_0^2}{\delta+\mu}+\eta_{0,x}^2+u_{0,x}^2\Big)dx\\ &\quad\quad+\frac{C}{\mu-\delta}\int_0^t\int_{\mathbb{R}}\big(\eta^2+\eta_x^2+u^2+u_x^2\big)dxds.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}(\eta^2+u^2+\eta_x^2+u_x^2)dx &\le C_1\int_{\mathbb{R}}(\eta_0^2+u_0^2+\eta_{0,x}^2+u_{0,x}^2)dx\\ &\quad+C_2\int_0^t\int_{\mathbb{R}}(\eta^2+\eta_x^2+u^2+u_x^2)dxds,\end{aligned}$$

where C_1, C_2 are positive constants independent of t. In consequence,

$$\int_{\mathbb{R}}(\eta^2+u^2+\eta_x^2+u_x^2)dx\le C_3+C_2\int_0^t\int_{\mathbb{R}}(\eta^2+\eta_x^2+u^2+u_x^2)dxds,$$

where $C_3=C_1\int_{\mathbb{R}}(\eta_0^2+u_0^2+\eta_{0,x}^2+u_{0,x}^2)dx$ is a constant independent of t. Then Gronwall's inequality implies that

$$\int_{\mathbb{R}}(\eta^2+u^2+\eta_x^2+u_x^2)dx\le C_3e^{C_2T_0},$$

for any $t \in [0, T_0]$. We have that (u, η) satisfies the following a priori bound in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, as long as it exists:

$$\|(u,\eta)\|_{H^1\times H^1} = \|u\|_{H^1} + \|\eta\|_{H^1} \leq C\|(u_0,\eta_0)\|_{H^1\times H^1}. \tag{10}$$

Here we used the fact that the norms

$$\|(u,\eta)\| = (\|u\|_{H^1}^2 + \|\eta\|_{H^1}^2)^{1/2}, \quad \text{and} \quad \|(u,\eta)\|_{H^1\times H^1} = \|u\|_{H^1} + \|\eta\|_{H^1}$$

are equivalent.

In order to estimate the norm of the solution (u, η) in $H^2(\mathbb{R})$, differentiating both equations in system (1) leads to

$$\eta_{tx} + \delta\eta_{txxx} + u_{xx} + \mu u_{txxx} + \theta u_{xxxx} + \alpha(u^p\eta)_{xx} = 0, \tag{11}$$

$$u_{tx} + \delta u_{txxx} + \eta_{xx} + \mu\eta_{txxx} + \theta\eta_{xxxx} + \alpha(u^p u_x)_x = 0. \tag{12}$$

Multiplying equation (11) by η_x and equation (12) by u_x and integrating with respect to the spatial variable x, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\eta_x^2 - \delta\eta_{xx}^2)dx + \int_{\mathbb{R}}(\eta_x u_{xx} + \mu\eta_x u_{txxx} + \theta u_{xxxx}\eta_x + \alpha(u^p\eta)_{xx}\eta_x)dx = 0,$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u_x^2 - \delta u_{xx}^2)dx + \int_{\mathbb{R}}(\eta_{xx} u_x + \mu\eta_{txxx} u_x + \theta\eta_{xxxx} u_x + \alpha(u^p u_x)_x u_x)dx = 0.$$

Summing the equations above, using integration by parts and applying the decay properties of the solution (u, η) at infinity, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\eta_x^2 + u_x^2 - \delta\eta_{xx}^2 - \delta u_{xx}^2 - 2\mu\eta_{xx}u_{xx})dx \\
&= -\alpha\int_{\mathbb{R}}\big((u^p\eta)_{xx}\eta_x + (u^p u_x)_x u_x\big)dx = \alpha\int_{\mathbb{R}}\big((u^p\eta)_x\eta_{xx} + u^p u_x u_{xx}\big)dx \\
&=\alpha\int_{\mathbb{R}}\big((pu^{p-1}u_x\eta + u^p\eta_x)\eta_{xx} + u^p u_x u_{xx}\big)dx \\
&\leq(\|u\|_\infty^{p-1}\|\eta\|_\infty\|u_x\|\|\eta_{xx}\| + \|u\|_\infty^p\|\eta_x\|\|\eta_{xx}\| + \|u\|_\infty^p\|u_x\|\|u_{xx}\|) \\
&\leq C(\|u_x\|^2 + \|u_{xx}\|^2 + \|\eta_x\|^2 + \|\eta_{xx}\|^2).
\end{aligned}$$

Integrating with respect to the time t in the interval $[0, t]$, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}}(\eta_x^2 + u_x^2 - \delta\eta_{xx}^2 - \delta u_{xx}^2)dx \\
&\quad\leq \int_{\mathbb{R}}(\eta_{0,x}^2 + u_{0,x}^2 - \delta\eta_{0,xx}^2 - \delta u_{0,xx}^2 - 2\mu\eta_{0,xx}u_{0,xx})dx+ \\
&\quad C\int_0^t\int_{\mathbb{R}}(u_x^2 + \eta_x^2 + u_{xx}^2 + \eta_{xx}^2)dxds + 2\mu\int_{\mathbb{R}}\eta_{xx}u_{xx}dx \\
&\quad\leq \int_{\mathbb{R}}(\eta_{0,x}^2 + u_{0,x}^2 - \delta\eta_{0,xx}^2 - \delta u_{0,xx}^2 - \mu\eta_{0,xx}^2 - \mu u_{0,xx}^2)dx+ \\
&\quad C\int_0^t\int_{\mathbb{R}}(u_x^2 + \eta_x^2 + u_{xx}^2 + \eta_{xx}^2)dxds - \mu\|\eta_{xx}\|^2 - \mu\|u_{xx}\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}} \eta_x^2 + u_x^2 + (\mu-\delta)\eta_{xx}^2 + (\mu-\delta)u_{xx}^2 dx \\ &\quad \leq \int_{\mathbb{R}} (\eta_{0,x}^2 + u_{0,x}^2 - (\delta+\mu)\eta_{0,xx}^2 - (\delta+\mu)u_{0,xx}^2)dx + \\ &\quad C\int_0^t \int_{\mathbb{R}} (u_x^2 + \eta_x^2 + u_{xx}^2 + \eta_{xx}^2)dxds.\end{aligned}$$

We conclude that there exist positive constants C_1, C_2 such that

$$\int_{\mathbb{R}} (\eta_x^2 + u_x^2 + \eta_{xx}^2 + u_{xx}^2)dx \leq C_1 + C_2 \int_0^t \int_{\mathbb{R}} (u_x^2 + \eta_x^2 + u_{xx}^2 + \eta_{xx}^2)dxds.$$

Using Gronwall's inequality, we obtain the estimate

$$\int_{\mathbb{R}} (\eta_x^2 + u_x^2 + \eta_{xx}^2 + u_{xx}^2)dx \leq C_1 e^{C_2 T_0},$$

for any $t \in [0, T_0]$. Thus we have that the norm of the solution (u, η) is bounded in $H^2(\mathbb{R}) \times H^2(\mathbb{R})$ by a constant independent of time t as long as it exists. Using a process of induction we establish that the norm $\|(u, \eta)\|_{H^m \times H^m}$, with $1 \leq m \leq s$ integer, is bounded by a constant independent of t. Thus we can apply the fixed point principle as done in [13] by using $\eta(x, T_0)$ and $u(x, T_0)$ as initial data for extending the solution to the interval $[T_0, 2T_0]$. Continuing in this way, a global solution to the Cauchy problem (1) can be defined. Solution uniqueness follows using Gronwall's inequality. □

4 Hamiltonian structure

We can rewrite equations (1) in the following manner:

$$\begin{pmatrix} I + \delta\partial_x^2 & \mu\partial_x^2 \\ \mu\partial_x^2 & I + \delta\partial_x^2 \end{pmatrix} \begin{pmatrix} \eta \\ u \end{pmatrix}_t = \begin{pmatrix} -u - \theta u_{xx} - \alpha u^p \eta \\ -\eta - \theta\eta_{xx} - \alpha\frac{u^{p+1}}{p+1} \end{pmatrix}_x . \qquad (13)$$

Here I denotes the identity operator. It can be shown that the operator $\tilde{\mathcal{A}} : H^2 \times H^2 \to L^2 \times L^2$, defined by

$$\tilde{\mathcal{A}} = \begin{pmatrix} I + \delta\partial_x^2 & \mu\partial_x^2 \\ \mu\partial_x^2 & I + \delta\partial_x^2 \end{pmatrix}$$

is invertible provided that the conditions on the parameters δ and μ in Theorem 3 are satisfied. Thus, system (13) can be written as

$$\begin{pmatrix} \eta \\ u \end{pmatrix}_t = \begin{pmatrix} I + \delta\partial_x^2 & \mu\partial_x^2 \\ \mu\partial_x^2 & I + \delta\partial_x^2 \end{pmatrix}^{-1} \partial_x \begin{pmatrix} -u - \theta u_{xx} - \alpha u^p \eta \\ -\eta - \theta\eta_{xx} - \alpha\frac{u^{p+1}}{p+1} \end{pmatrix}.$$

Defining the Hamiltonian

$$\mathcal{H}(\eta,u)=\frac{1}{2}\int_{\mathbb{R}}\big(\theta\eta_x^2+\theta u_x^2-\eta^2-u^2-2\alpha\frac{u^{p+1}\eta}{p+1}\big)dx, \tag{14}$$

we obtain the Hamiltonian structure for system (1):

$$\begin{pmatrix}\eta\\ u\end{pmatrix}_t=\begin{pmatrix}I+\delta\partial_x^2 & \mu\partial_x^2\\ \mu\partial_x^2 & I+\delta\partial_x^2\end{pmatrix}^{-1}\begin{pmatrix}0 & \partial_x\\ \partial_x & 0\end{pmatrix}\begin{pmatrix}\mathcal{H}_\eta\\ \mathcal{H}_u\end{pmatrix}. \tag{15}$$

We observe that the Hamiltonian in (14) is formally conserved in time along solutions of system (1). This can be established as follows:

$$\begin{aligned}&\frac{d}{dt}\mathcal{H}(\eta,u)=\\ &\quad\frac{1}{2}\int_{\mathbb{R}}\big(2\theta\eta_x\eta_{xt}+2\theta u_xu_{xt}-2\eta\eta_t-2uu_t-2\alpha u^pu_t\eta-\frac{2\alpha}{p+1}u^{p+1}\eta_t\big)dx.\end{aligned}$$

Using integration by parts and the fact that a solution (η,u) of system (1) decays to zero at infinity, we obtain that

$$\frac{d}{dt}\mathcal{H}(\eta,u)=-\int_{\mathbb{R}}\big[\big(\theta\eta_{xx}+\eta+\frac{\alpha}{p+1}u^{p+1}\big)\eta_t+\big(\theta u_{xx}+u+\alpha u^p\eta\big)u_t\big]dx.$$

Then taking into account the fact that

$$\begin{aligned}&\int_{\mathbb{R}}\Big[(\delta u_{xt}+\mu\eta_{xt})\eta_t+(\delta\eta_{xt}+\mu u_{xt})u_t\Big]dx=\\ &\qquad\int_{\mathbb{R}}\big(\delta(u_t\eta_t)_x+\frac{1}{2}\mu(\eta_t^2)_x+\frac{1}{2}\mu(u_t^2)_x\big)dx=0,\end{aligned}$$

and using equations (1) we realize that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(\eta,u)=\int_{\mathbb{R}}\Big[&(\theta\eta_{xx}+\eta+\frac{\alpha}{p+1}u^{p+1})(\delta\eta_{xt}+\mu u_{xt}+u+\theta u_{xx}+\alpha u^p\eta)_x+\\ &(\theta u_{xx}+u+\alpha u^p\eta)\Big(\delta u_{xt}+\mu\eta_{xt}+\eta+\theta\eta_{xx}+\alpha\frac{u^{p+1}}{p+1}\Big)_x+\\ &(\delta u_{xt}+\mu\eta_{xt})\eta_t+(\delta\eta_{xt}+\mu u_{xt})u_t\Big]dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\mathcal{H}(\eta,u)=\\ &\int_{\mathbb{R}}\Big[\big(\theta\eta_{xx}+\eta+\frac{\alpha u^{p+1}}{p+1}+\delta u_{xt}+\mu\eta_{xt}\big)\big(\theta u_{xx}+u+\alpha u^p\eta+\delta\eta_{xt}+\mu u_{xt}\big)\Big]_x dx\\ &=0.\end{aligned}$$

5 Existence of solitary wave solutions

We are interested in determining whether system (1) has travelling wave solutions which can be written in the special form:

$$u(x,t) = u(\xi), \qquad \eta(x,t) = \eta(\xi),$$

where $\xi = x - ct$, the parameter c is a constant denoting the wave speed and

$$(\eta^{(n)}(\xi), u^{(n)}(\xi)) \to (0,0),$$

as $|\xi| \to \infty$, for $n = 0, 1, 2, 3, ...$ This class of solution is called **a solitary wave**. Its existence is possible due to the balance between the effects of nonlinear and dispersive terms present in the system.

It is important to note that in general, we cannot find analytic expressions for a solitary wave of the full system (1). Further note that the functions $u(\xi), \eta(\xi)$ must satisfy the following non-trivial system of ordinary differential equations (solitary wave equations):

$$\begin{aligned} &(\mu c - \theta)u'' + c\delta\eta'' - u + c\eta - \alpha(u^p\eta) = 0, \\ &c\delta u'' + (\mu c - \theta)\eta'' + cu - \eta - \alpha(\frac{u^{p+1}}{p+1}) = 0, \end{aligned} \tag{16}$$

where the derivatives are with respect to ξ. A curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is called a homoclinic orbit around the origin of system (16) if it satisfies $\gamma(\xi) \to (0,0)$ and $\gamma'(\xi) \to (0,0)$ as $|\xi| \to \infty$. A very important observation which enables us to study existence of travelling wave solutions of system (1) is that a homoclinic solution around the origin (0,0) of system (16) leads to a solitary wave solution of system (1) due to the decay properties at infinity of a solitary wave.

There are some analytical results with regard to the existence of homoclinic solutions of systems which are cast in the form of (16). In particular, in this paper, we will use the following result due to Toland [18] which provides necessary conditions that guarantee existence of homoclinic orbits for a special kind of system formed by two second-order ordinary differential equations:

Theorem 4. *Denote* $\overrightarrow{u} = (u(\xi), \eta(\xi))^T$ *and consider a system in the form*

$$S_1\overrightarrow{u}'' + S_2\overrightarrow{u} + \nabla g(u,\eta) = 0, \tag{17}$$

where S_1 *and* S_2 *are* 2×2 *symmetric matrices and the derivatives are with respect to* ξ, $g \in C^2(\mathbb{R}^2, \mathbb{R})$ *satisfies that* $g, \nabla g$ *and the second partial derivatives of* g *are all zero at the origin* $(0,0)$. *Define* $Q(\overrightarrow{u}) = \overrightarrow{u}^T S_1 \overrightarrow{u}$ *and* $f(u,\eta) = \overrightarrow{u}^T S_2 \overrightarrow{u} + 2g$. *In addition, let us assume that*

(I) $\det(S_1) < 0$, *so* Q *is indefinite and there exist two linearly independent vectors* $\overrightarrow{u}_1 = (u_1, \eta_1)^T$ *and* $\overrightarrow{u}_2 = (u_2, \eta_2)^T$ *where* Q *vanishes.*

(II) There exists a closed curve F *which passes through the origin of* $\mathbb{R}^2$ *such that*

(i) $f = 0$ *on* F*, and* $F - \{(0,0)\}$ *lies in the set* $\{(u,\eta) : Q(u,\eta) < 0\}$*,*

(ii) $f(u,\eta) > 0$ *in the (non-empty) interior of* F*,*

(iii) $F - \{(0,0)\}$ *is strictly convex, i.e.,*

$$D = f_{uu} f_\eta^2 - 2 f_{u\eta} f_u f_\eta + f_{\eta\eta} f_u^2 < 0$$

on $F - \{(0,0)\}$*,*

(iv) $\nabla f(u,\eta) = 0$ *on* F *if and only if* $(u,\eta) = (0,0)$*.*

Then there exists an orbit $\xi \to \gamma(\xi) = (u(\xi), \eta(\xi))^T$ *of system (17) which is homoclinic to the origin and which has the following properties:*

(a) $(u(0), \eta(0)) \in \hat{F}$*, where* $\hat{F}$ *is the segment between* P_1 *and* P_2*, with* P_i *satisfying* $\nabla f(P_i).\overrightarrow{u}_i = 0$ *for* $i = 1, 2$*.*

(b) $\gamma(\xi)$ *is an even solution, i.e.,* $\gamma(\xi) = \gamma(-\xi)$*,* $\gamma'(\xi) = -\gamma'(-\xi)$ *for all* $\xi \in \mathbb{R}$*.*

(c) $\gamma(\xi)$ *is in the interior of the curve* F *for all* $\xi \in \mathbb{R} - \{0\}$*.*

(d) γ *is monotone in the sense that* $\gamma(\xi) \leq \gamma(s)$ *if* $\xi \geq s \geq 0$*.*

In the next theorem we summarize our results concerning solitary wave solutions of system (1).

Theorem 5. *Suppose that* $\delta < \mu \leq 0$*,* $\theta > 0$*,* $\alpha > 0$*,* $p \geq 1$*, and* $c > max\{1, \frac{\theta}{\mu - \delta}\}$*. Then the Boussinesq system (1) has an even solitary wave solution* $u(x,t) = u(x - ct)$*,* $\eta(x,t) = \eta(x - ct)$*.*

Proof. Our strategy to establish the existence of solitary wave solutions of system (1) will be to show that it is in the setting of the previous theorem and satisfies all of its hypotheses.

In the first place, system (16) can be cast in the form of (17) with $\overrightarrow{u} = (u,\eta)^T$,

$$S_1 = \begin{pmatrix} \mu c - \theta & c\delta \\ c\delta & \mu c - \theta \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & c \\ c & -1 \end{pmatrix},$$

and

$$g(u,\eta) = -\frac{\alpha}{p+1} u^{p+1} \eta.$$

We define

$$Q(\overrightarrow{u}) = \overrightarrow{u}^T S_1 \overrightarrow{u} = (\mu c - \theta) u^2 + 2 c\delta u\eta + (\mu c - \theta)\eta^2,$$

and

$$f(u,\eta) = \overrightarrow{u}^T S_2 \overrightarrow{u} + 2g = -u^2 + 2cu\eta - \eta^2 - \frac{2\alpha}{p+1} u^{p+1} \eta.$$

Observe that $\det(S_1) < 0$ since $c > \frac{\theta}{\mu-\delta}$. Thus, the quadratic form Q is indefinite and

$$\overrightarrow{u}_1 = \begin{pmatrix} \frac{-c\delta+\sqrt{c^2\delta^2-(\mu c-\theta)^2}}{\mu c-\theta} \\ 1 \end{pmatrix}, \quad \overrightarrow{u}_2 = \begin{pmatrix} \frac{-c\delta-\sqrt{c^2\delta^2-(\mu c-\theta)^2}}{\mu c-\theta} \\ 1 \end{pmatrix}$$

are linearly independent vectors (located in the quadrant II) where Q vanishes. Moreover, Q is negative in quadrants I and III since $c > \theta/\mu$ (if $\mu < 0$) and $\delta < 0$.

Let us define the curve $F = \{(u,\eta) \in \mathbb{R}^2 : f(u,\eta) = 0\}$. We will show that this curve satisfies all hypotheses (II) in Theorem 4. Obviously, $(0,0) \in F$. Fix $0 < u_0 < \big(\frac{(c-1)(p+1)}{\alpha}\big)^{1/p}$. Then in the plane (u,η), the vertical line $u = u_0$ crosses the curve F only at the two points $(u_0, \eta_\pm)$, where

$$\eta_\pm = cu_0 - \frac{\alpha}{p+1}u_0^{p+1} \pm \sqrt{\Big(cu_0 - \frac{\alpha}{p+1}u_0^{p+1}\Big)^2 - u_0^2}.$$

It is easy to see that $\eta_\pm$ belongs to $\mathbb{R}$, $\eta_\pm \geq 0$, and $\eta_- < \eta_+$. Therefore F is a closed curve which lies in the first quadrant, where the quadratic form Q is negative. Now we can factor the function $f(u,\eta)$ as

$$f(u,\eta) = -(\eta - \eta_+)(\eta - \eta_-).$$

Thus, we see that $f(u,\eta) > 0$ at the points (u,η) located in the interior of the curve F, since η is between η_- and η_+. We have verified the hypotheses (II) (i) and (ii) in Theorem 4.

To show that the set $F - \{(0,0)\}$ is strictly convex, we study the sign of the quantity

$$D = f_{uu}f_\eta^2 - 2f_{u\eta}f_u f_\eta + f_{\eta\eta}f_u^2.$$

To do this, first note that

$$f_u = -2u + 2c\eta - 2\alpha u^p\eta = \frac{-2u^2 + 2c\eta u - 2\alpha u^{p+1}\eta}{u}$$

and using the equation $2c\eta u = u^2 + \eta^2 + \frac{2\alpha u^{p+1}\eta}{p+1}$ (i.e. $f = 0$), we obtain

$$f_u = \frac{\eta^2 - u^2 - \frac{2\alpha p}{p+1}u^{p+1}\eta}{u}.$$

Analogously, we obtain

$$f_\eta = 2cu - 2\eta - \frac{2\alpha u^{p+1}}{p+1} = \frac{2cu\eta - 2\eta^2 - \frac{2\alpha}{p+1}u^{p+1}\eta}{\eta} = \frac{u^2 - \eta^2}{\eta}.$$

$$f_{u\eta} = 2c - 2\alpha u^p = \frac{2cu\eta - 2\alpha u^{p+1}\eta}{u\eta} = \frac{u^2 + \eta^2 - \frac{2\alpha p}{p+1}u^{p+1}\eta}{u\eta},$$

and

$$f_{\eta\eta} = -2, \qquad f_{uu} = -2 - 2\alpha p u^{p-1}\eta.$$

Then substituting the expressions above in the coefficient D, we arrive at

$$\begin{aligned} D =& (-2 - 2\alpha p u^{p-1}\eta)\Big(\frac{u^2-\eta^2}{\eta}\Big)^2 \\ & - 2\Big(\frac{u^2+\eta^2-\frac{2\alpha p u^{p+1}\eta}{p+1}}{u\eta}\Big)\Big(\frac{\eta^2-u^2-\frac{2\alpha p u^{p+1}\eta}{p+1}}{u}\Big)\Big(\frac{u^2-\eta^2}{\eta}\Big) \\ & - 2\Big(\frac{\eta^2-u^2-\frac{2\alpha p u^{p+1}\eta}{p+1}}{u}\Big)^2. \end{aligned}$$

After some algebra and simplifying the expression above, we obtain that

$$D = -\frac{2p\alpha u^{p-1}}{(p+1)^2\eta}\big((p+1)^2(\eta^2-u^2)^2 + 4\alpha p\eta u^{p+3}\big) < 0,$$

in the whole first quadrant (since $u > 0, \eta > 0, \alpha > 0, p \geq 1$), and therefore the coefficient D is also negative along the set $F - \{(0,0)\}$.

Finally, suppose that $\nabla f(u,\eta) = 0$ on the curve F. Then the point (u,η) satisfies the system

$$-u + c\eta - \alpha u^p \eta = 0, \tag{18}$$

$$cu - \eta - \frac{\alpha}{p+1}u^{p+1} = 0 \tag{19}$$

$$-u^2 + 2cu\eta - \eta^2 - \frac{2\alpha}{p+1}u^{p+1}\eta = 0. \tag{20}$$

Then from equation (19)

$$-\frac{2\alpha}{p+1}u^{p+1}\eta = -2cu\eta + 2\eta^2.$$

Therefore substituting in equation (20) we obtain that $\eta^2 - u^2 = 0$, i.e. $\eta = \pm u$. If $\eta = u$ then equations (18) and (19) imply

$$\begin{aligned} u(c-1) - \alpha u^{p+1} &= 0, \\ u(c-1) - \frac{\alpha}{p+1}u^{p+1} &= 0. \end{aligned}$$

Therefore, $\frac{p}{p+1}u^{p+1} = 0$, and thus $u = 0$ and $\eta = 0$.

On the other hand, if $\eta = -u$, again from equations (18) and (19), we obtain

$$\begin{aligned} u(c+1) - \alpha u^{p+1} &= 0, \\ u(c+1) - \frac{\alpha}{p+1}u^{p+1} &= 0. \end{aligned}$$

We conclude one more time that $u = 0$ and $\eta = 0$.

Thus we have verified hypothesis (II) (iv). By Theorem 4, we conclude that the second-order equations (16) have a homoclinic solution around the origin $(0, 0)$ satisfying the properties (a)-(d), and in consequence, the Boussinesq-type system (1) has an even solitary wave solution for the wave speed $c > \max\{1, \frac{\theta}{\mu-\delta}\}$. □

6 Approximation of solitary wave solutions

In this section we will derive some approximations of the solitary wave solutions of system (1) whose existence was established in the previous section for the wave speed $c > \max\{1, \frac{\theta}{\mu-\delta}\}$, with $\theta > 0$, $\alpha > 0$ and $\delta < \mu \leq 0$. We also assume that these parameters are small.

Observe that the system for solitary wave solutions (16) can be rewritten in a more convenient form as:

$$\alpha(u^p\eta) = B\eta - Au, \quad \frac{\alpha u^{p+1}}{p+1} = Bu - A\eta,$$

where A and B are second-order linear operators defined as $A = I - (\mu c - \theta)\partial_x^2$ and $B = c(I + \delta\partial_x^2)$. Applying the operators A and B to the previous equations respectively, it follows that

$$\alpha A(u^p\eta) = AB\eta - A^2u, \quad \alpha B\frac{\alpha u^{p+1}}{p+1} = B^2u - BA\eta.$$

Adding the equations above, we obtain that

$$\alpha(A(u^p\eta) + B\frac{u^{p+1}}{p+1}) = (B^2 - A^2)u. \tag{21}$$

We point out the following approximations:

$$B^2 = c^2(I + 2\delta\partial_x^2) + O(\delta^2), \quad A^2 = I - 2(\mu c - \theta)\delta\partial_x^2) + O(\mu^2, \theta^2, \mu\theta).$$

Because of $\alpha Bu - \alpha A\eta = \frac{\alpha^2 u^{p+1}}{p+1} = O(\alpha^2)$, we deduce that

$$\alpha c(I + \delta\partial_x^2)u - \alpha(I - (\mu c - \theta)\partial_x^2)\eta = O(\alpha^2),$$

$$\alpha cu - \alpha\eta = O(\alpha\delta, \alpha\mu, \alpha\theta),$$

from which it follows that

$$\eta = cu + O(\alpha, \theta, \mu, \delta). \tag{22}$$

Observe also that

$$\alpha A = \alpha I - \alpha(\mu c - \theta)\partial_x^2 = \alpha I + O(\alpha\mu, \alpha\theta)$$
$$\alpha B = \alpha c(I + \delta\partial_x^2) = \alpha cI + \alpha c\delta\partial_x^2 = \alpha cI + O(\alpha\delta).$$

Using these facts and substituting the approximations for the operators A and B into equation (21), we come to

$$\alpha u^p\eta+\frac{\alpha c u^{p+1}}{p+1}=(c^2-1)u+2(\delta c^2+(\mu c-\theta))u''+\text{second-order terms in }\alpha,\theta,\mu,\delta.$$

After neglecting second-order terms in α,δ,θ,μ, and using equation (22), we get that

$$\alpha c\Big(\frac{p+2}{p+1}\Big)u^{p+1}+(1-c^2)u-2(\delta c^2+\mu c-\theta)u''=0. \tag{23}$$

We remark that equation (23) corresponds to the equation of a solitary wave solution with velocity $\tilde{c}=c^2$ of the generalized KdV equation

$$u_t+u_x+\gamma_1 u^p u_x+\gamma_2 u_{xxx}=0,$$

where

$$\gamma_1=\alpha c(p+2),\quad \gamma_2=-2(\delta c^2+\mu c-\theta).$$

Analogously the wave elevation η satisfies up to second-order terms in the parameters $\alpha,\beta,\theta,\delta,\mu$ the generalized KdV equation

$$\eta_t+\eta_x+\frac{\gamma_1}{c^p}\eta^p\eta_x+\gamma_2\eta_{xxx}=0.$$

This fact shows a close relationship between the solitary wave solutions of system (1) and the generalized KdV equation.

Multiplying by u' the equation above, using the decay property of u at infinity and integrating, we arrive at the first-order ordinary differential equation

$$(u')^2=\frac{\alpha u^{p+2}}{c(p+1)(\delta+\mu/c-\theta/c^2)}+\frac{(1-c^2)u^2}{2c^2(\delta+\mu/c-\theta/c^2)}. \tag{24}$$

It is well known that equation (24) has solutions in the form

$$u(\xi)=A\ \mathrm{sech}^{2/p}(B\xi), \tag{25}$$

where

$$A=\Big(\frac{c(c^2-1)(p+1)}{2\alpha c^2}\Big)^{1/p},\qquad B=\frac{p}{2c}\sqrt{\frac{1-c^2}{2(\delta+\mu/c-\theta/c^2)}}.$$

We point out that since $c>0>\theta/\mu$, we have that $\theta-\mu c>0>\delta c^2$ which implies that $\delta+\mu/c-\theta/c^2<0$ and thus the constants A,B are well defined. To compute the wave elevation η, one can use the approximation

$$\eta(\xi)=cu(\xi). \tag{26}$$

In Figure 1 we plot the curve

$$F=\left\{(u,\eta)\in\mathbb{R}^2: f(u,\eta)=-u^2+2cu\eta-\eta^2-\frac{2\alpha}{p+1}u^{p+1}\eta=0\right\},$$

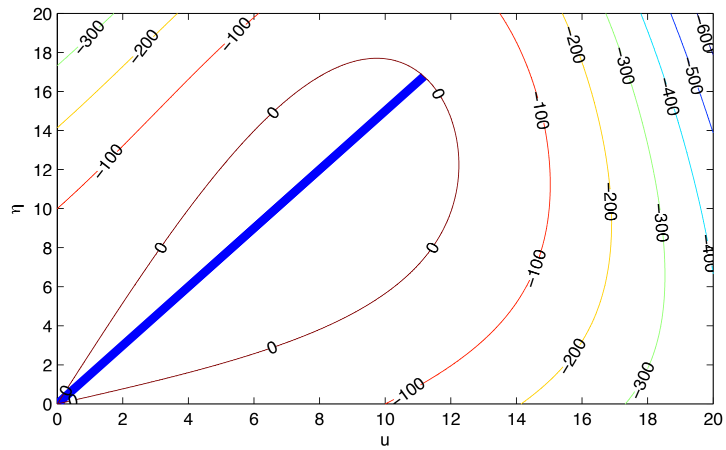

Figure 1: We plot in darker line a homoclinic curve of system (16) (i.e. a solitary wave of system (1)). The thin lines correspond to some level curves of the function $f(u, \eta) = -u^2 + 2cu\eta - \eta^2 - \frac{2\alpha}{p+1} u^{p+1}\eta$.

given in Theorem 5 for the parameters $\theta = 0.01, \mu = -0.01$, $\delta = -0.03$, $c = 1.5$, $p = 2$, $\alpha = 0.01$, together with the approximation of a solitary wave curve $\{(u, \eta)\}$ given in equations (25) and (26). We see that this figure is consistent with all of the theoretical results presented in the previous section.

7 Conclusions

In this paper we extended the results in [13] about local existence of solutions for the Cauchy problem (1). We also found the Hamiltonian structure of this system, which could be useful for establishing future results on the properties of its solutions and for designing numerical solvers which conserve this Hamiltonian. It is well-known that schemes that preserve conserved quantities of the model equations possess better error propagation mechanisms that their nonconservative counterparts. From the physical point of view, it is also important to know conserved functionals of a system. Finally, in certain range of the wave velocity and modelling parameters we showed the existence of solitary wave solutions of system (1). The strategy was to use a result by Toland [18] and the ideas by Chen [8] regarding homoclinic orbits of a system formed by two second-order ordinary differential equations.

References

[1] Amick, C.J. ; Toland, J.F. Global uniqueness of homoclinic orbits for a class of fourth order equations. Z. Angew Math. Phys. **43** (1992), 591-597.

[2] Bona, J.L. ; Chen, M. ; Saut, J.C. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media I: Derivation and the linear theory. J. Nonlinear Sci. **12** (2002), 283-318.

[3] Bona, J.L.; Chen, M.; Saut, J.C. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media II: The nonlinear theory. Nonlinearity **17** (2004), 925-952.

[4] Bona, J. ; Chen, H. Solitary waves in nonlinear dispersive systems. Discrete and Continuous Dynamical Systems, series B, **2**,(3): (2002), 313-378.

[5] Boussinesq, J.V. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. J. Math. Pures Appl. **17** (1872), 55-108.

[6] Champneys, A.R. ; Kuznetsove, Y.A. ; Sandstede, B. A numerical toolbox for homoclinic bifurcation analysis. Internat. J. Bifur. Chaos Appl. Sci. Eng. **6** (1996), 867-887.

[7] Champneys, A.R.; Spence, A. Hunting for homoclinic orbits in reversible systems: A shooting technique. Advances in Computational Mathematics **1** (1993), 81-108.

[8] Chen, M. Solitary wave and multipulsed traveling-wave solutions of Boussinesq systems. Applicable Analysis **75** (2000), 213-240.

[9] Kivshar, Y. S.; Agrawal, G. Optical Solitons: From Fibers to Photonic Crystals. Academic Press, 1 Ed., United Sates, 2003.

[10] Madsen, P.A.; Murray, R. ; Sørensen, O.R. A new form of the Boussinesq equations with improved linear dispersion characteristics, Part 1. Coastal Eng. **15** (1991), 371-388.

[11] Madsen, P.A.; Sørensen, O.R. A new form of the Boussinesq equations with improved linear dispersion characteristics, Part 2, A slowly-varying bathymetry. Coastal Eng. **18** (1992), 183-204.

[12] Muñoz, G., J.C. ; Nachbin, A. Improved Boussinesq-type equations for highly variable depth. IMA J. Appl. Math. **71** (2006), 600-633.

[13] Muñoz, G., J.C. ; Rivas, I. Existence and Asymptotic decay for a Boussinesq-type Model. Journal of Mathematical Analysis and Applications **345** (2008), 455-475.

[14] Nachbin, A. A terrain-following Boussinesq system. SIAM J. Appl. Math. **63** (2003), 905-922.

[15] Nwogu, O. Alternative form of Boussinesq equations for nearshore wave propagation. J. Waterwave, Port Coast. Ocean Eng. **119** (1993), 618-638.

[16] Peregrine, D.H. Long waves on a beach. J. Fluid Mech. **27** (1967), 815-827.

[17] Schäffer, H.A.; Madsen, P.A. Further enhancements of Boussinesq-type equations. Coastal Eng. **26** (1995), 1-14.

[18] Toland, J.F. Existence of symmetric homoclinic orbits for systems of Euler-Lagrange equations. Proceedings of Symposia in Pure Mathematics **45** (1986), 447-459.

Authors' address
Juan Carlos Muñoz Grajales — Departamento de Matemáticas, Universidad del Valle, Cali-Colombia
e-mail: `jcarlmz@yahoo.com`
Ivonne Rivas Triviño — IMPA, Rio de Janeiro, Brazil
e-mail: `ivonriv@gmail.com`